Quiz 7, Linear Algebra

Fall 2017, Dr. Adam Graham-Squire

6 min ⇒ 25 min

Name: Key

1. (3 points) Let V and W be vector spaces and $T : V \to W$ a linear transformation. Let H be a nonzero subspace of V, then $T(H)$ is a subspace of W given by the images of vectors in H. Prove that dim $T(H) \leq$dim H.

$\dim T(H) =$ # of basis vectors for $T(H)$

Let $n = \dim(H) =$ # of basis vectors for H.

Let $\{v_1, v_2, \ldots, v_n\}$ be basis for H.

Claim: $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ spans $T(H)$.

Let $g \in T(H)$. Then $g = T(\vec{h})$ for some vector $\vec{h} \in H$, and $\vec{h} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ since $\{v_i\}$ are a basis

$\Rightarrow T(\vec{h}) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$ by linearity

$$g = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

so g is in the span of $\{T(v_1), \ldots, T(v_n)\}$

$\Rightarrow$ ~~Basis~~ dimension for $T(H)$ must be less than or equal to n, since n vectors span $T(H)$.

$\Rightarrow \dim T(H) \leq \dim(H)$

2. (3 points) Determine the dimensions of Nul A and Col A for $A = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Briefly explain your reasoning.

dim (Col A) = # pivot columns = 3

dim (Nul A) = # free variables = 2

3. (4 points) Use an inverse matrix to find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$.

$x = P_\beta [x]_\beta$ 0.5

$P_\beta = \begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}$

$\Rightarrow P_\beta^{-1} \vec{x} = [x]_\beta$

$P_\beta^{-1} = \frac{1}{5-6} \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$

$P_\beta^{-1} = \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}$ ✓ 0.5

$\Rightarrow$ ✓ $\begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = [x]_\beta$

✓ $\begin{bmatrix} 5 \\ 1 \end{bmatrix} = [x]_\beta$